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INVOLUTIONS IN BANACH ALGEBRA

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ABSTRACT : In this paper, we define some definition related to involution, examples, Gelfand Nahnark Theorem has important role.

1.INTRODUCTION : Involution has important role in Banach algebra. In this paper we discuss self adjoint, hermitian, Banach algebra corollary, Gelfand Nahnark Thoeorem, isometry and isomorphism studied.

2.Definition :A map $x \to x^*$ of a complex algebra A into A is called an invoution of A if it has the following properties for all $x, y \in A$ and $\lambda \in C$.

(1) $(x+y)^* = x^* + y^*$

(2) $(\lambda x)^* = \bar{\lambda} x^*$

(3) $(xy)^* = y^* x^*$

(4) $x^{**} = x$

2.1.Definition : If $x \in A$ and $x^* = x$, then x is called hermitian or self adjoint.

Example : $f \to \bar{f}$ is an involution on C(X)

2.1.Theorem : If A is a Banach algebra with an involution, and if $x \in A,$, then

a) $x + x^*, i(x - x^*)$ and xx^* are hermitian.

b) x has a unique representation x = u+iv where $u, v \in A$ and u and v are hermitian.

c) The unit element e is hermitian

d) x is invertible in A if and only if x* is invertible in which case $(x^*)^{-1} = (x^{-1})^*$ and

e) $\lambda \in \sigma(x)$ iff $\bar{\lambda} \in \sigma(x^*)$

Proof: $(x+x^*) = x^* + x^{**} = x^* x = x + x^*$. Hence $x + x^*$ is hermitian.

$[i(x-x^*)]^* = \bar{i}\,(x-x^*)^* = -i[x^* - (x^*)^*] = -i(x^* - x) = i(x - x^*)$

$(xx^*)^* = (x^*)^* \cdot x^* = x \cdot x^*$. Hence $i(x-x)^*$ and x^* and x* are hermitian.

a) Put $u = \frac{x+x^*}{2}$ and $v = \frac{i(x^*-x)}{2}$. Then x=u+iv. Clearly u, v are hermitian since $x + x^*$ is hermitian and also $\frac{i(x^*-x)}{2}$ is hermitian. The uniqueness of the representation is yet to be proved. If $u` + iv` = x$ is another representation then put $w = v` - v.$. Then both w and iw are hermitian and $iw = (iw)^* = -iw^* = -iw$ i.e. $iw + iw = 0$ i.e. $2iw = 0$ (ie) $v` = v.$. Since $v` = v, u` = u$

Hence the representation is unique.

b) Clearly $e^* = ee^*.$. But ee^* is self adjoint. Hence e* is self adjoint. Hence e is self adjoint.

c) Since x is invertible $\exists, x^{-1}\, s.t.\, x x^{-1} = e$

Now $(x x^{-1})^* = (x^{-1}) x^* = e^* = e (\because$ e is self adjoint)

$\therefore (x^{-1})^*$ is the inverce of x*

But $(x^*)^{-1}$ is the inverse of x* and hence $(x^{-1}) = (x^*)^{-1}$

d) Let $\lambda \in \sigma(x)$. Then $(\lambda e - x)$ is not invertible. Hence $(\lambda e - x)^*$ is not invertible (ie) $(\bar{\lambda} e - x^*)$ is not invertible. Hence $\bar{\lambda} \in \sigma(x^*)$ the converse follows analogously.

3.Definition : If A is a Banach algebra with an involution *, which satisfies the $\|xx^*\| = \|x\|^2$ for every $x \in A$ then A is called a B* algebra.

3.1.Theorem : If A is a semi simple commulative Banach algebra, then involution on A is continuous

Proof: Let h be a homomorphism of A

Define $\emptyset(x) = \bar{h}(x^*)$,

Then $\emptyset(x+y) \quad = \bar{h}[x+y]^* \quad = \bar{h}(x^*+y^*)$

$$= \bar{h}(x^*) + \bar{h}(y^*) \qquad = \emptyset(x) + \emptyset(y)$$

$$\emptyset(\alpha.x) = \bar{h}[(\alpha,x)^*] \quad = \bar{h}(\bar{\alpha}x^*)$$

$$= \overline{h(\bar{\alpha},x^*)} = \overline{\bar{\alpha}h(x^*)}$$

$$= \alpha \cdot \bar{h}(x^*)$$

$$= \alpha\phi(x)$$

Similarly $\emptyset(xy) = \bar{h}((xy)^*) \quad = \bar{h}(y^*x^*)$

$$= \overline{h(y^*x^*)}$$

$$= \overline{h(y^*)h(x^*)}$$

$$= \bar{h}(y^*)\bar{h}(x^*)$$

$$= \emptyset(y).\emptyset(x) = \emptyset(x).\emptyset(y)$$

Hence Ø is a complex homomorphism on A. Then Ø is continuous. For, suppose $x_n \to x,$ and $x_n^* \to y$ in A

Then $\bar{h}(x^*) = \emptyset(x) = Lim\emptyset(x_n) = Lim\bar{h}(x_n^*) = \bar{h}(y)$

This is true for every $h \in \Delta.$

Since A is semisimple x* = y. Hence $x \to x^*$ is continuous by closed graph theorem.

3.1.Corollary

A is a B* algebra, iff $\|x^*\| = \|x\| \forall x \in A$

and $\|xx^*\| = \|x\|\|x\|^*$

For, we have $\|x\|^2 = \|xx^*\| \leq \|x\|\|x^*\|$

Hence $\|x\| \leq \|x^*\|$ .... (1)

Similarly $\|x^*\| \leq \|x^{**}\| = \|x\|$ .... (2)

From (1) and (2) $\|x^*\| = \|x\|$

Now $\|xx^*\| = \|x\|^2 = \|x\| \cdot \|x\| = \|x\|\|x^*\|$

Conversely, we have that if $\|x\| = \|x^*\|$ for every $x \in A$ and $\|xx^*\| = \|x\| \cdot \|x^*\|$, then $\|xx^*\| = \|x\|\|x^*\| = \|x\| \cdot \|x\| = \|x\|^2$. Hence A is a B*-Algebra

3.2. Theorem: Gelfand-Nahnark Theorem

Suppose A is a commulative B* algebra, with maximal ideal space Δ.. The Gelfand transform is then an isometric isomorphism of A onto $C(\Delta)$ which has the additional property that

$$h(x^*) = \overline{h(x)} (x \in A, h \in \Delta)$$

or equivalently, that

$$(x^*)\hat{} = \overline{\hat{x}} (x \in A)$$

In particular, x is hermitian if and only if $\hat{x}$ is a real function.

The above theorem is called Gelfand - Nahnark theorem

Proof: Let $u \in A s.t. u = u*$. Let $h \in \Delta$. We have to prove that h(u) is real.

Put z=u+ite for real t. If $h(u) = \alpha + i\beta$ where α, β are reals then

$$h(z) = h(u + ite) = h(u) + h(ite)$$

$$= \alpha + i\beta + it.h(e)$$

$$= \alpha + i\beta + it = \qquad \alpha + i(\beta + t)$$

$$zz* = u^2 + t^2 e \text{ so that}$$

$$\alpha^2 + (\beta + t)^2 = |h(z)|^2 \le \|z\|^2 = \|zz*\| \le \|u\|^2 + t^2$$

or $\alpha^2 + \beta^2 + 2\beta t \le \|u\|^2 \quad \forall t \in$ Real

But this implies that $\beta = 0.$ = 0. Hence h(u) is real.

If $x \in A$, then $x = u + iv$ with $u = u*, v = v*$

Hence $x* = u - iv.$. Since $\hat{u}$ and $\hat{\vartheta}$ are real, we have

$$(x*)^{\wedge}(h) = \hat{u} - i\hat{v}](h) \text{ for every } h \in \Delta (i.e.)(x*)^{\wedge} = \overline{\wedge_x}$$

Thus $\bar{A}$ is closed under complex conjugation. By Stone Weierstrass theorem is dense in $C[\Delta]$

If $x \in A$ and $y = x x*$, then y = y*. Hence $\|y^2\| = \|y\|^2$. By induction, we get that llymi $\|y^m\| = \|y\|^m$ for every $m = 2^n$.

Hence $\|\hat{y}\|_\infty = \|y\|$ by the spectral radius formula. Since y = xx*

we have $\hat{y} = \hat{x}(x*\hat{)} = \hat{x}\bar{\hat{x}} = |\hat{x}|^2$

Hence $\|\hat{x}\|_{\infty}^{2} = \|y\| = \|xx^*\| = \|x\|^2$ or $\|\hat{x}\|_{\infty} = \|x\|$..

Thus $x \to \hat{x}$ is an isometry. Hence $\bar{A}$ is closed in $C(\Delta)$. Since A is also dense in $C(\Delta)$, we conclude that $\hat{A} = C(\Delta)$.. Hence the proof.

3.3.Theorem : If A is a commutative B* algebra which contains an element *x* such that the polynomials in *x* and *x** are dense in A. then the formula $(\Psi f)^{\wedge} = f\, o\, \hat{x}$ defines an isometric isomorphism Ψ of $C(\sigma(x))$ onto A which statisfies.

$\Psi\hat{f} = (\Psi f)^*$ for every $f \in C(\sigma(x))$. More over if $f(\lambda) = \lambda$ on $\sigma(x)$ then $\Psi f = x$.

Proof: Let Δ be the maximal ideal space of A. We know that $\hat{x}$ is a continuous function on Δ. The range of $\hat{x}$ is $\sigma(x)$. Suppose $h_1, h_2 \in \Delta$ and $\hat{x}(h_2)$, then $h_1(x) = h_2(x)$ and hence $h_1(x^*) = h_2(x^*)$ by the previous theorem. If P is any polynomial in x and x*, then $h_1(P) = h_2(P)$ since h_1, and h_2, are homomorphisms. By hypothesis, the elements of the form P(x,x*) are dense in A and since h_1, and h_2, are continuous we have $h_1(y) = h_2(y)$ for every $y \in A$. Hence $h_1 = h_2$. Hence we have proved that $\hat{x}(h_1) = \hat{x}(h_2)$ for every $y \in A$. Hence $h_1 = h_2$, (i.e.) $\hat{x}$ is 1-1. Since $\hat{x}$ is continuous and onto $\sigma(x)$ and $|-|$ we have that *x* is homeomorphism of Δ onto $\sigma(x)$ (By Vadiyanatha swamy's theorem). The mapping $f \to f\, o\, \hat{x}$ is therefore an isometric isomorphism of $C(\sigma(x)) \to C(\Delta)$ which preserves complex conjugation.

By the previous theorem, each fox is the Gelfand tranform of a unique elements of A, which we denote by $\Psi\, f$ and which satisfies $\|\Psi f\| = \|f\|_{\infty}$ [Since we have that $(x^*)^{\wedge} = \hat{x}$ by the previous theorem]. We have $\Psi\hat{f} = (\Psi(f))^*$.. If $f(\lambda) = \lambda$, then $f\, o\, \hat{x} = \hat{x}$ so that we have $(\Psi\hat{f}) = \hat{x}(ie)\Psi f = x$.

We are interested in knowing the existence of square roots in a Banach algebra. The following theorem is one in that direction.

3.4.Theorem : Suppose A is a commutative Banach algebra with an involution. If *x* is a self adjoint element of A and if $\sigma(x)$ contains no real number $\lambda \le 0$, then there exists $y \in A$ with $y^2 = x$ and y = y*.

Proof: Let R denote the non positive real numbers and let $\Omega = C - R^-$.. There exists a holomorphic function $f \in H(\Omega)$ such that $f^2(\lambda)$ and f(1) = 1. Since $\sigma(x) \subset \Omega$, we can define $y \in A$ as

$$y = \hat{f}(x) = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda e - x)^{-1} d\lambda$$

Where Γ is any contour that surrounds $\sigma(x)$ in Ω. Then it can be proved that y^2 = x [For a proof the student is referred to Defn and theorem of "Functional Analysis" by Rudin. This is the required y and y* = y. To prove y*=y we need what is called Runge's theorem in complex analysis.

Since Ω is simply connected 'Runges'. Theorem gives polynomials P, that converge to funiformly on compact subsets of Ω. Define Q_n, by

$2Q_n(\lambda) = P_n(\lambda) + \overline{P_n(\lambda)}$. Since $f(\bar{\lambda}) = \overline{f(\lambda)}$ the polynomials $Q_n \to f$ in the same manner [(ie) uniformly on compact sets]

Define $y_n = Q_n(x).(n = 1, 2, 3, ...)$ By definition, the polynomials Q_n have real coefficients. Since x = x*, if follows that y_n, = y_n*

The element $y = \lim_{n\to\infty} y_n$, and hence y = y*. if f* is continuous. Even if f* is not assumed to be continuous we can give a different argument to prove that y=y* as follows.

Let R be the radical of A. Let $\pi : A \to A \setminus R$ be the quoteint map. Define an involution in A/R by

$[\pi(a)]^* = \pi(a^*)$ for $a \in A$

If a is heremitian, then so is it $\pi(a)$

Since π it is continuous, $\pi(y_n) \to \pi(y)$

Since A/R is isomorphic to A. A/R is semi simple and therefore every involution in A/R is continuous. Hence $\pi(y)$ is hermitian. Hence $\pi(y - y^*) = 0$ (ie) $y - y^*$ is in the radical of A.

Now we can write y = u + iv where u and v are hermitian. Since $y - y^* \in R$, hermitian v belongs to the radical of A. Since x = y^2 we have $x = u^2 - v^2 + 2iuv$..

Let h be a complex homomorphism on A. Since v is in the radical of A, h(v) = 0. Hence $h(x) = [h(u)]^2$ By luypothesis $0 \notin \sigma(x)$. Hence $h(x) \neq 0$. Hence $h(x) \neq 0$. This is true for every $h \in \Delta$(ie)u is invertible.

Since x=x*

and since $x = u^2 - v^2 + 2iuv$, we have that uy = 0

Since $v = u^{-1}(u\,v)$ we have that $v = u^{-1}.0 = 0$

Hence v = u and hence y* = u* But u is hermitian and hence y* = y

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